Threshold models for changing social norms in dynamic networks

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Abrupt change in social norms

Chinese footbinding

- Caused serious pain and health risks to women
- Was widespread for thousands of years
- May have begun as patriarchal practice to "ensure paternity confidence" and persisted as a simple coordination problem (Mackie 1996)
- Abandoned in a single generation

Female Genital Cutting?

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Other possible cases of abrupt social change

- Behaviors related to HIV transmission?
- Sanitation practices in traditional communities
- The Arab Spring and related social movements

Social norm as coordination problem

- individuals wouldn't do it alone
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Threshold model for coordination problems

A very simple model

- Personal beliefs, interests, access to resources, social roles, circumstances, etc. all influence people's tendency to **adopt** or **reject** a convention.
- Probably the simplest model for social norm: idealize them all in a **threshold**.
- adopt if current level of adoption exceeds the threshold; reject if not. (Schelling 1971; Granovetter 1978)

This simple model can have quite complex behavior, so it's worth understanding along with more complicated models!

Threshold model for coordination problems

With **contact networks** it's not well understood.

- People aren't influenced equally by everyone everywhere: they have a limited number of contacts.
- Consider spread of adoption/rejection on lattices and networks.
- With and without mixing.

Statistical physics models

Majority Rule (MR)

- all thresholds $T_i = 0.5$
- equivalent to **classical Ising model** (Krapivsky 2003)

$$\mathcal{H} = -\frac{1}{2} \sum_{i;j \in \langle i \rangle} s_i s_j \quad \textbf{Update rule}: \; s_i \mapsto \mathrm{sign} \sum_{j \in \langle i \rangle} s_j$$

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- thresholds' PDF $f(\cdot)$ *e.g.* Gaussian
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$$\mathcal{H} = -\sum_{i;j \in \langle i \rangle} \frac{s_i s_j}{2 n_i} + \sum_i (2T_i - 1) s_i \quad \textbf{Update rule}: \; s_i \mapsto \mathrm{sign}(y_i - T_i)$$

(where $y_i = \frac{1}{n_i} \sum_{j \in \langle i \rangle} \frac{1+s_j}{2}$ – proportion of plus spins in the i-th nhood)

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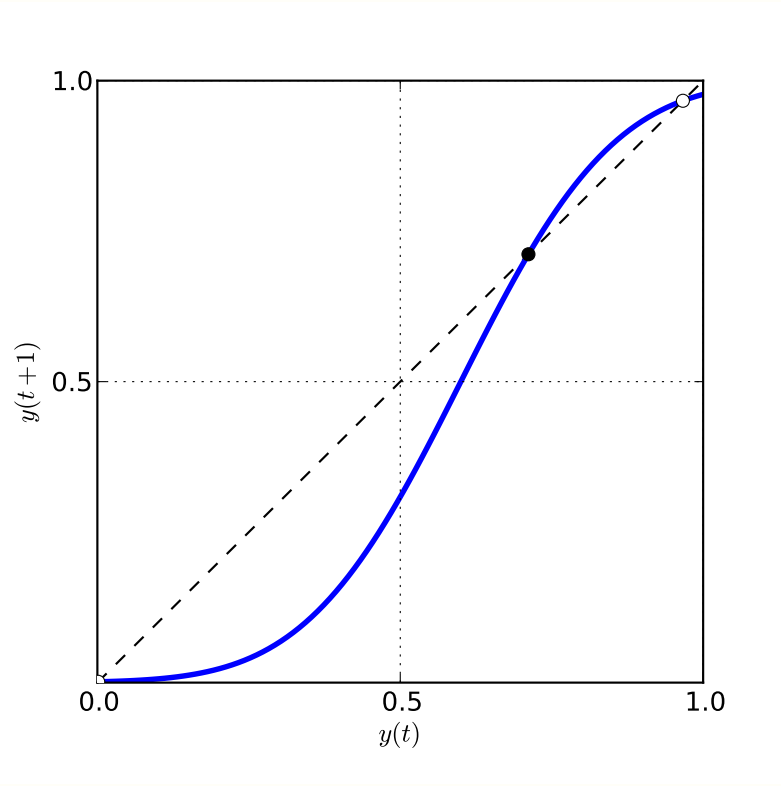
Infinitely large neighborhood



$y(t+1) = F(y(t))$

- $F(y) = \int_{-\infty}^{y} f(\xi)\,\mathrm{d}\xi$ – thresholds' CDF

Fig: thresholds' PDF is Gaussian, mean 0.6, s.d. 0.2

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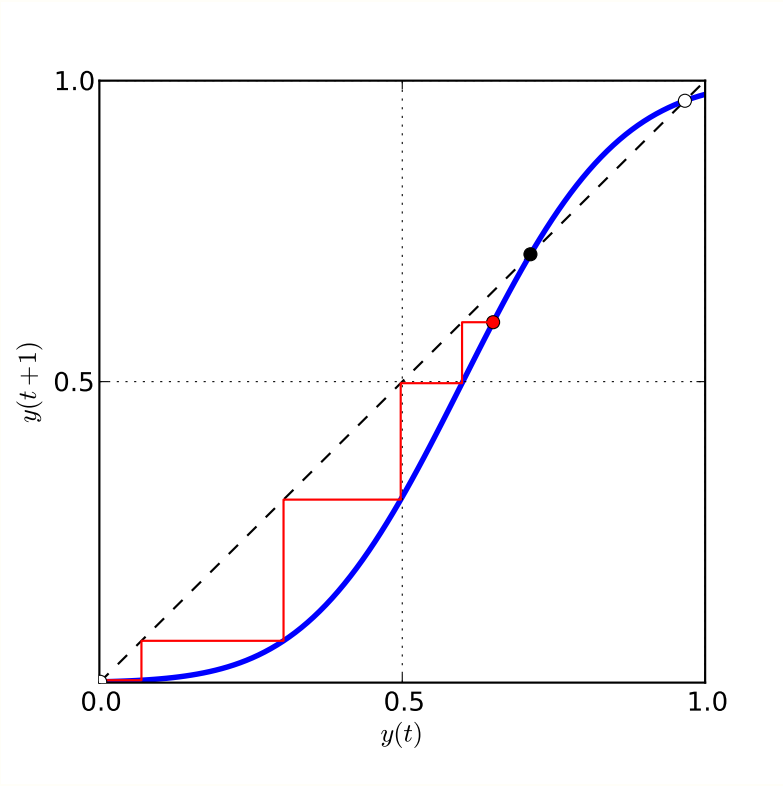

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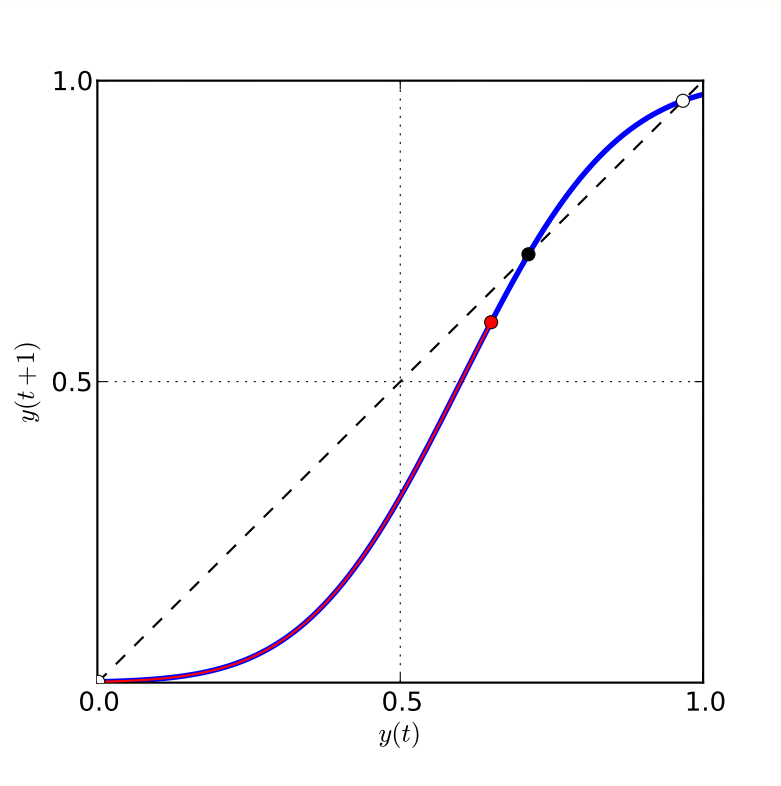

$$\frac{\mathrm{d}y}{\mathrm{d}t} = F(y) - y, \quad y(0) = y_0$$

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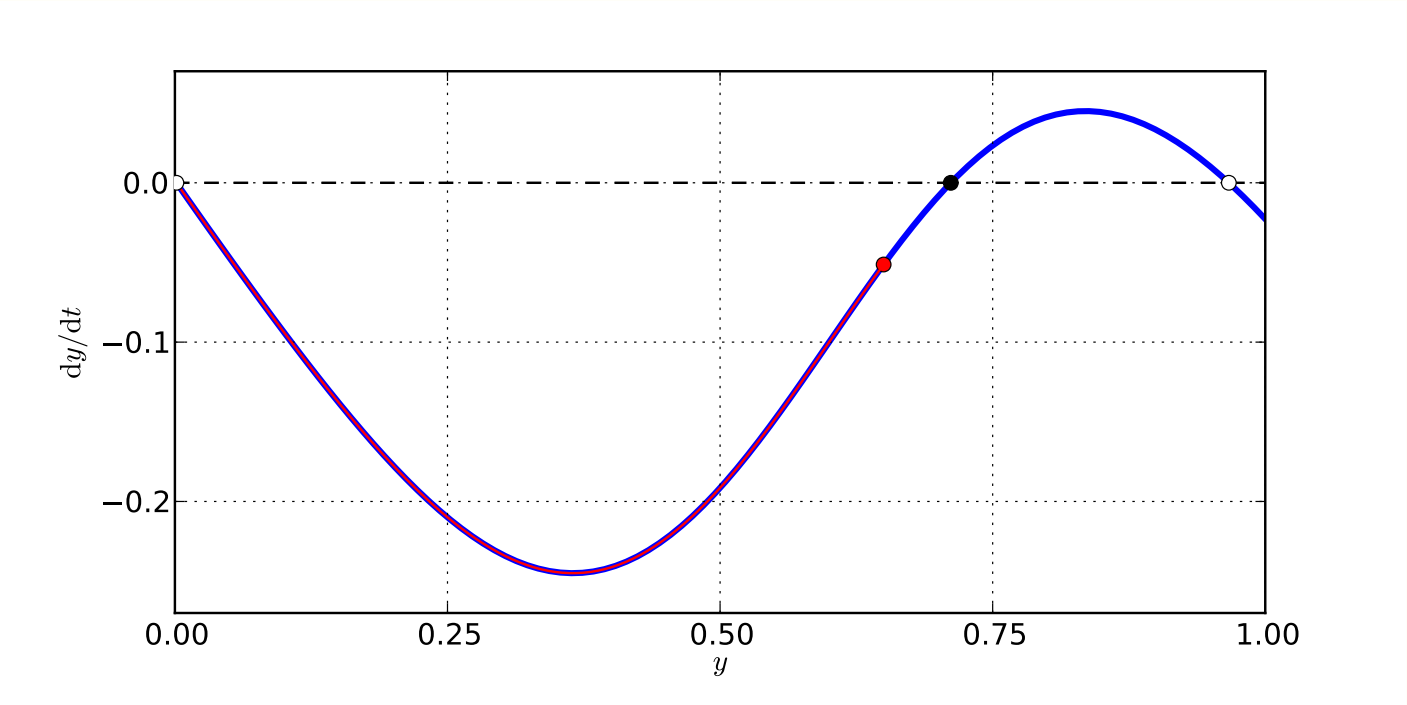

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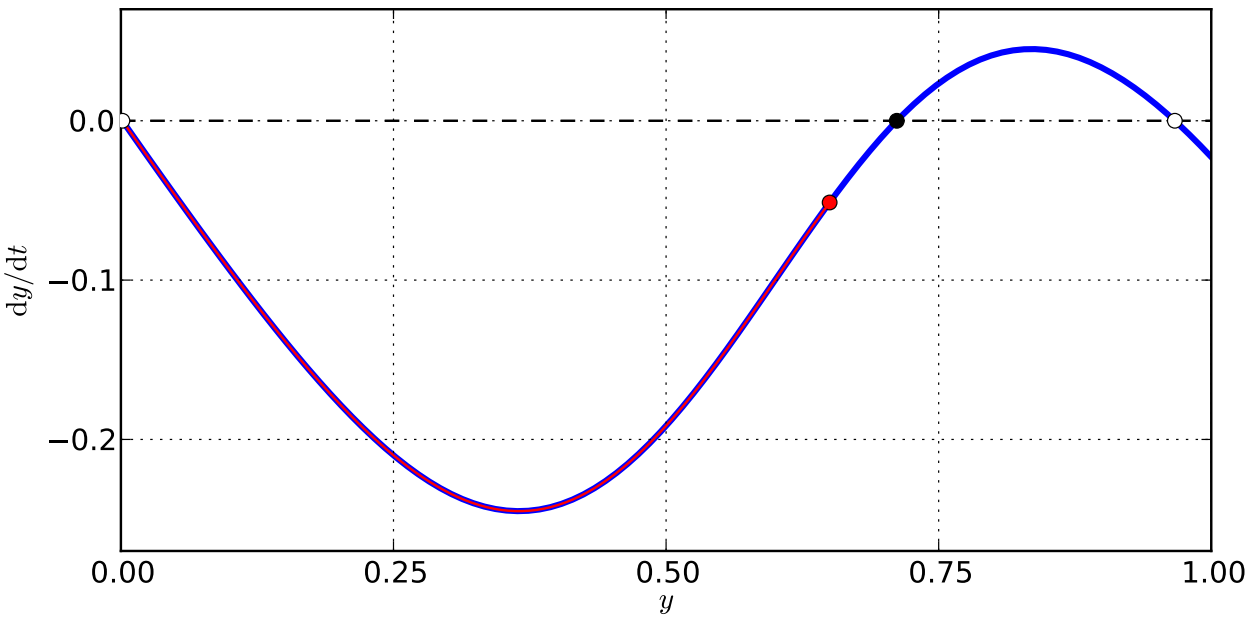
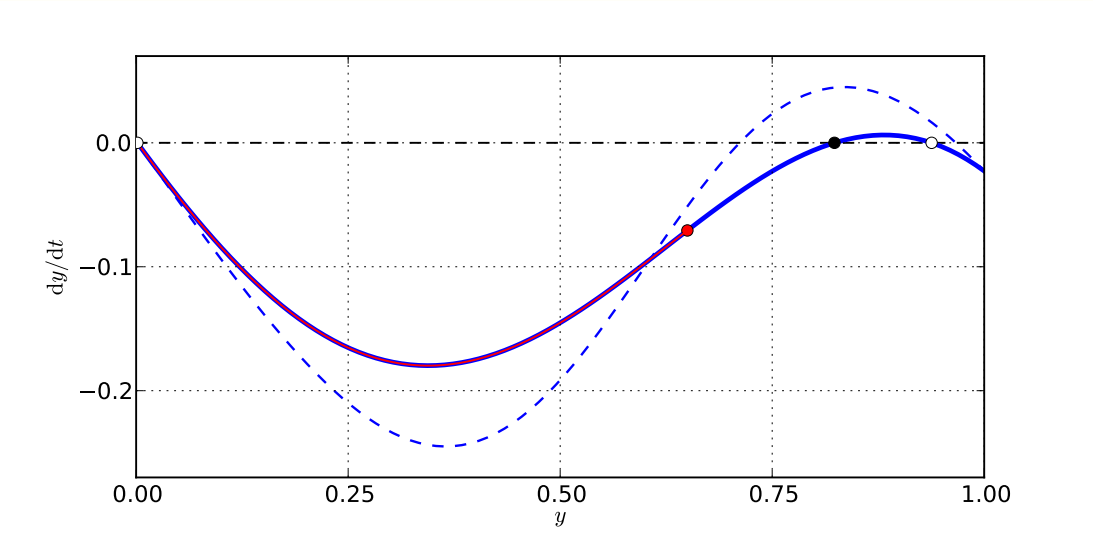


- It has a low-adoption equilibrium and a high-adoption one.
- Noise or change in threshold distribution could cause abrupt change from one to the other.

Fig: thresholds' PDF is Gaussian, mean 0.6, s.d. 0.2

Fast mixing ($\mu \gg 1$) limit with neighborhood structure ($N \gg 1$)



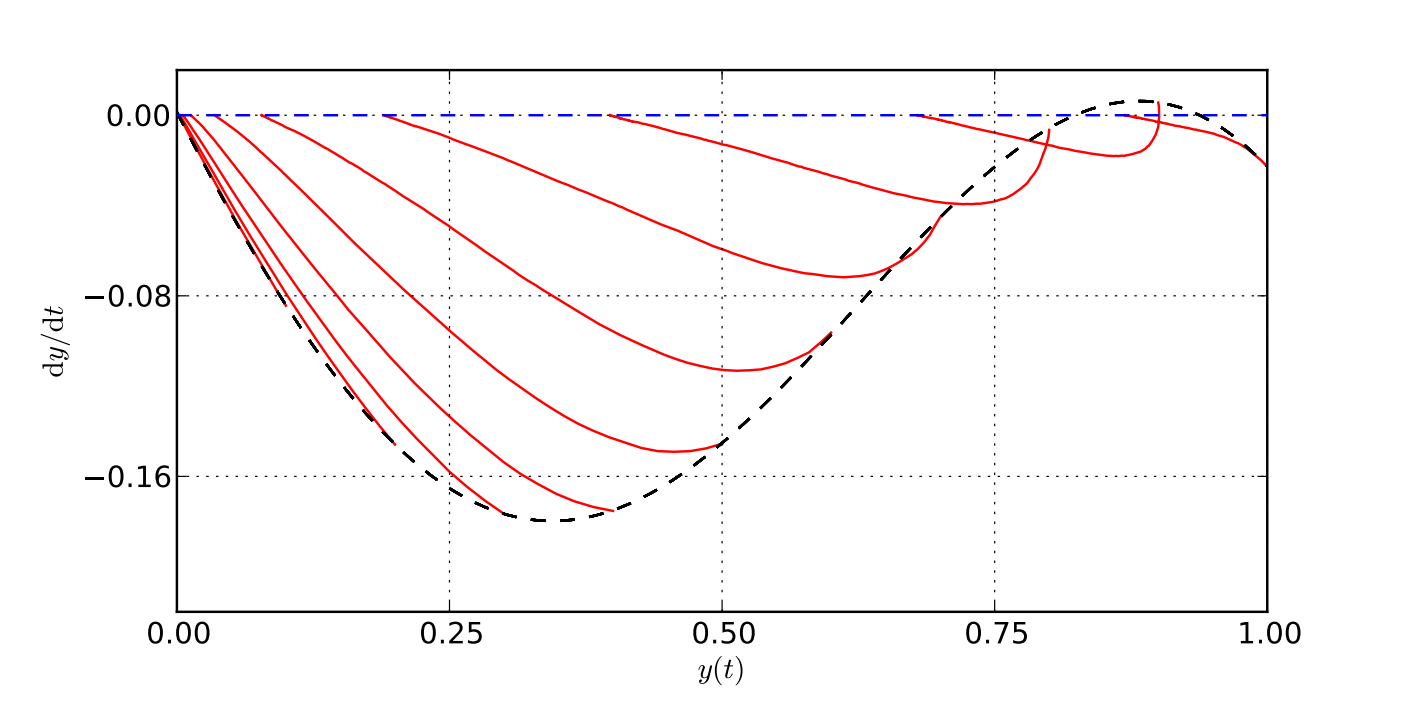
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \hat{F}(y) - y, \quad y(0) = y_0$$

where $\hat{F}(y) = \sum_{n=0}^{\infty} \mathcal{P}(n) F(y|n)$ such that

- $P(n)$ is the degree distribution of the network
- $F(y|n) = \sum_{k=0}^{n} F(\frac{k}{n}) C_n^k y^k (1-y)^{n-k}$ (C_n^k – binomial coeff.)

Fig: 2d lattice, nhood 8; thresholds' PDF is Gaussian, mean 0.6, s.d. 0.2

No mixing $\mu = 0$ with neighborhood structure, $N \gg 1$



Many **metastable equilibria** (Rosinberg 2008)

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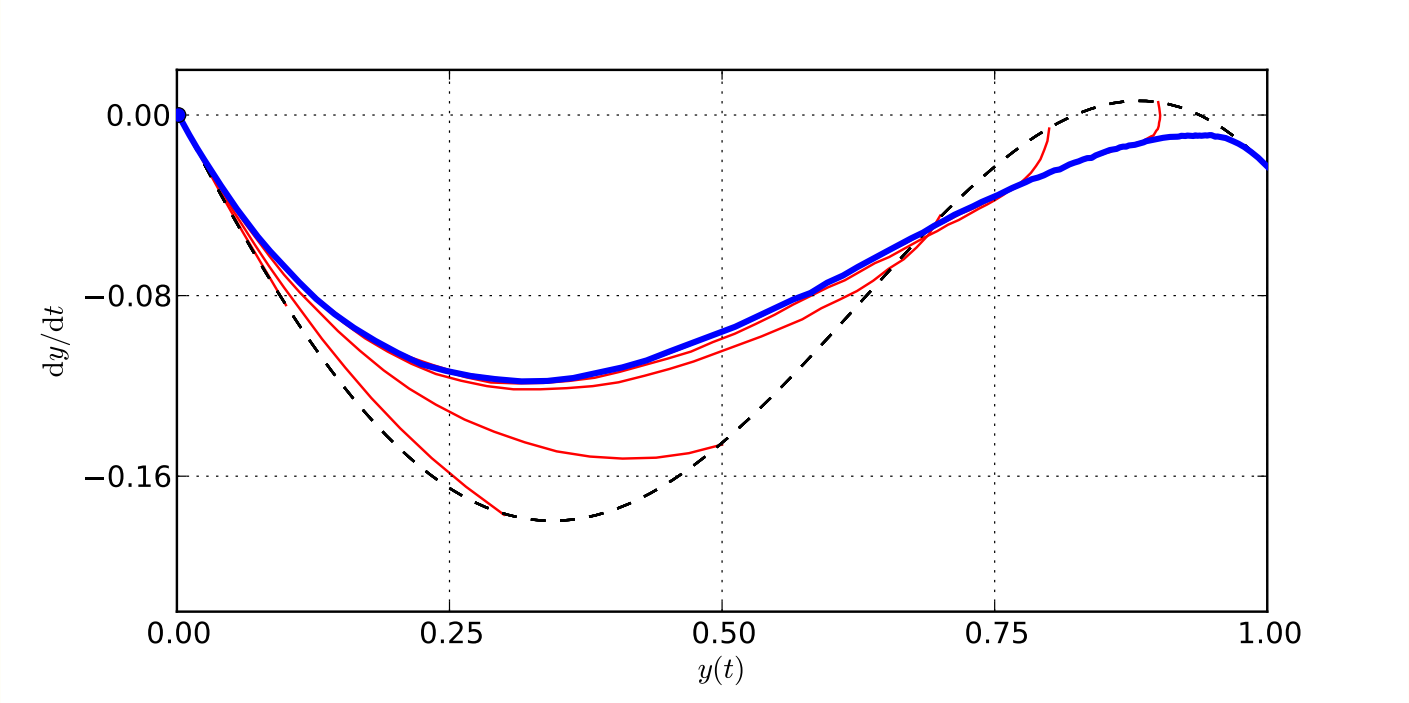
Intermediate mixing $\mu \sim 1$, large-scale limit $N \gg 1$

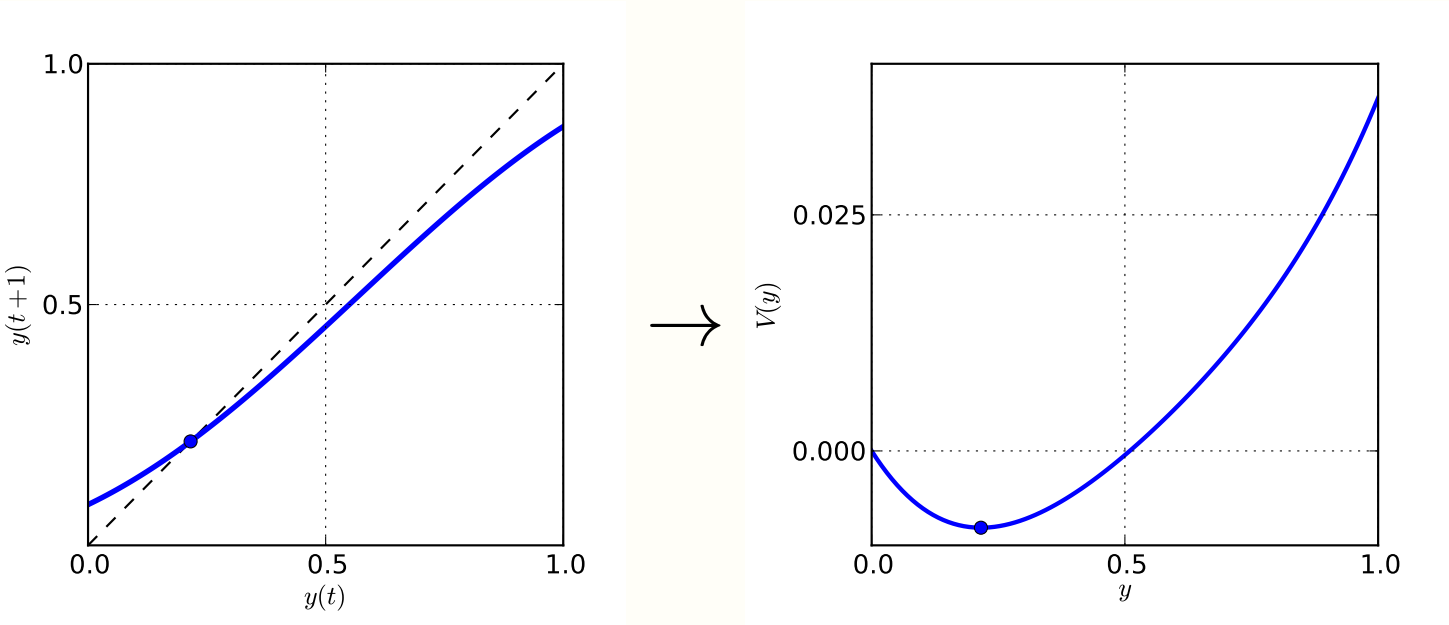


"Slow manifold" exists

Fig: 2d lattice, nhood 8; thresholds' PDF is Gaussian, mean 0.6, s.d. 0.2; $\mu = 0.25$

Potential function

$$V(y) = -\int^{y} (\hat{F}(\xi) - \xi)\,\mathrm{d}\xi, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -V'(y)$$



- 2d lattice, nhood 8; thresholds' PDF is Gaussian, mean 0.55, s.d. **0.4**
- Globally stable equilibrium

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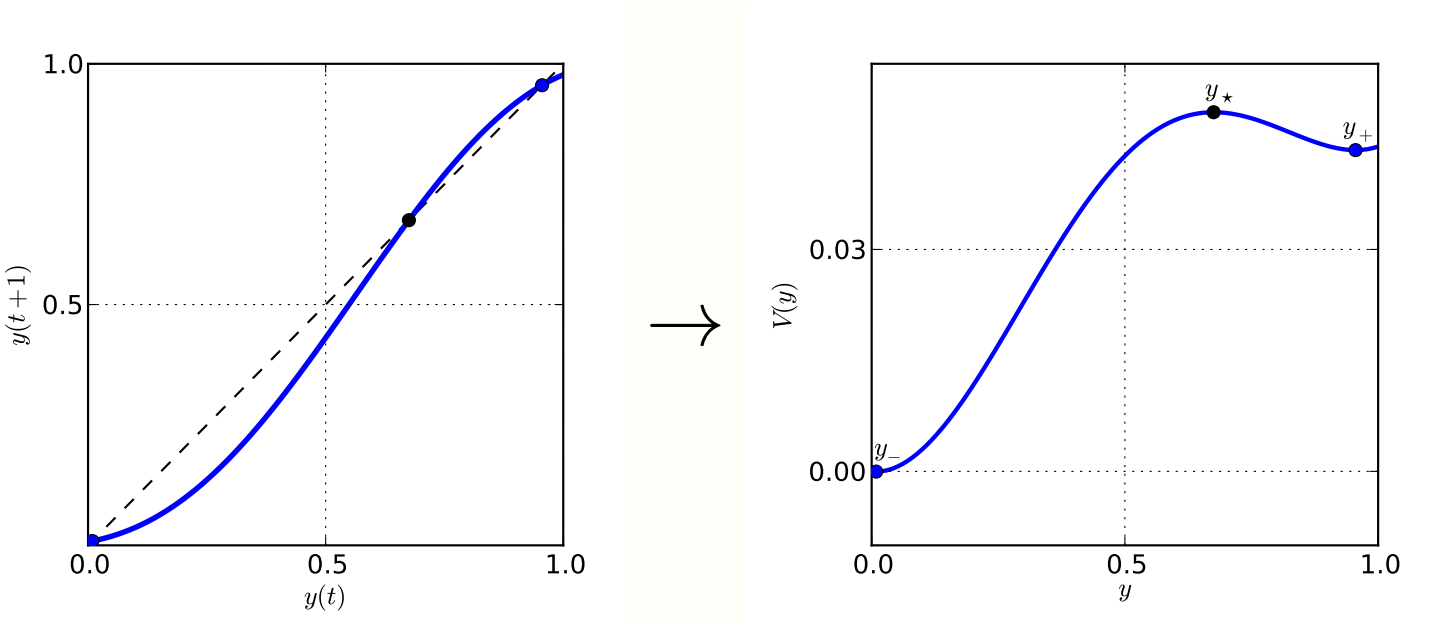
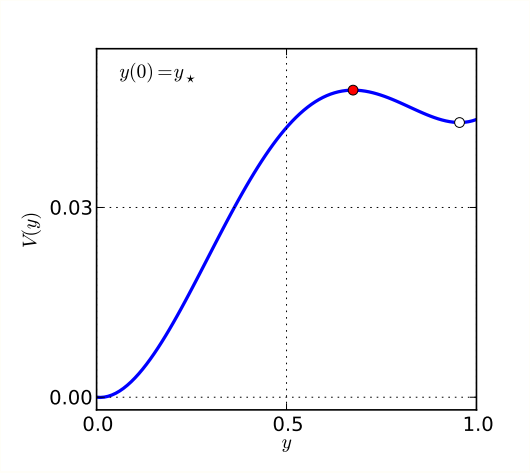
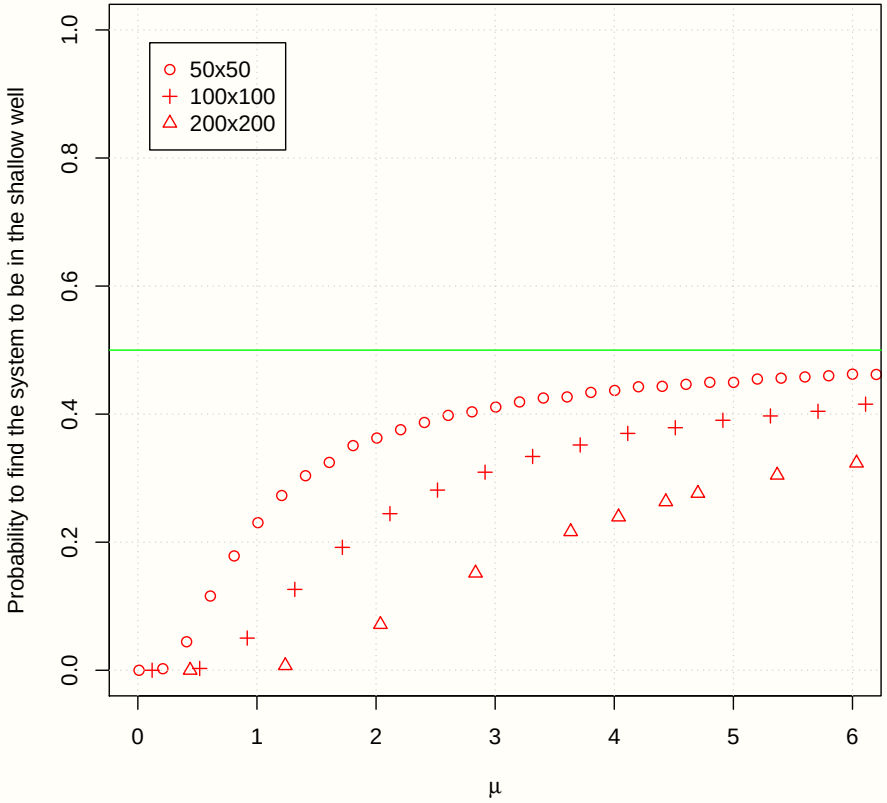


- 2d lattice, nhood 8; thresholds' PDF is Gaussian, mean 0.55, s.d. **0.225**
- $y_\pm$ – **stable**, $y_\star$ – **unstable**

$(\mu, L) \mapsto \mu/L$





- 2d lattice $L \times L$, nhood 8; thresholds' PDF is Gaussian, mean 0.55, s.d. 0.225; $y(0) = y_\star \approx 0.6752$
- 10^5 different initial realizations per each point

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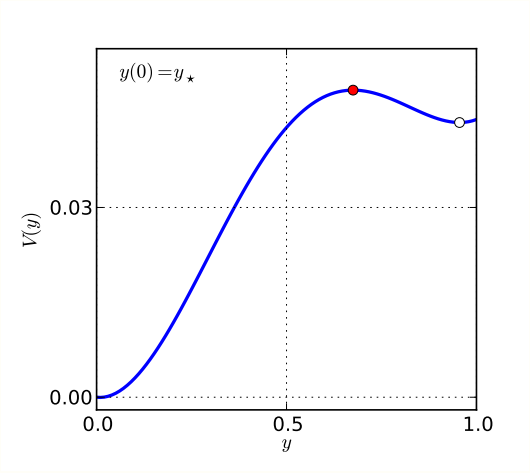

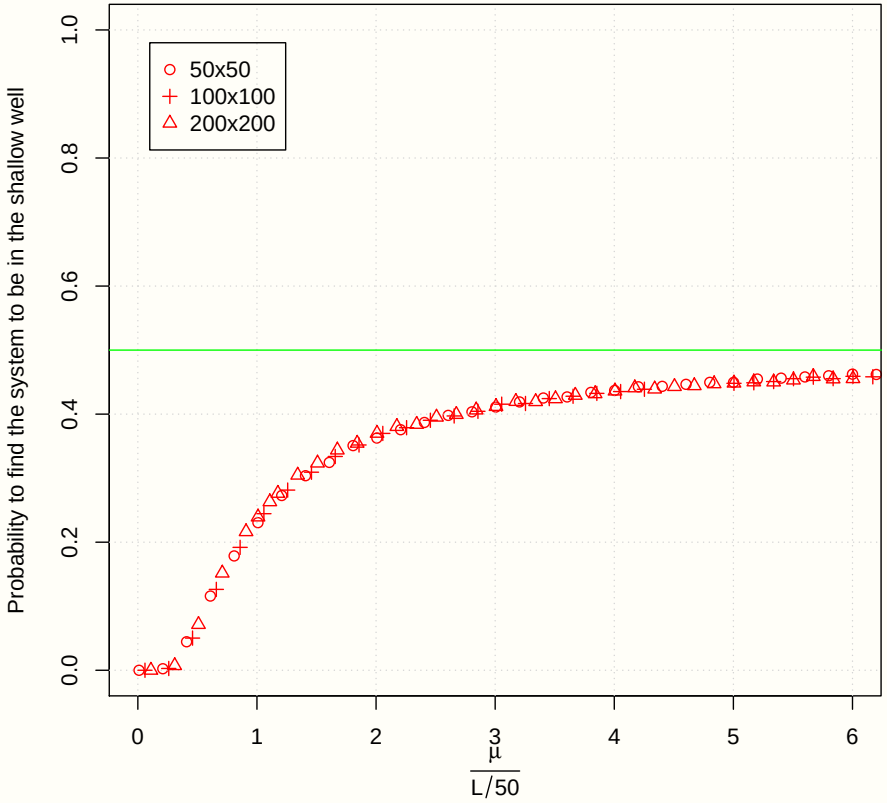

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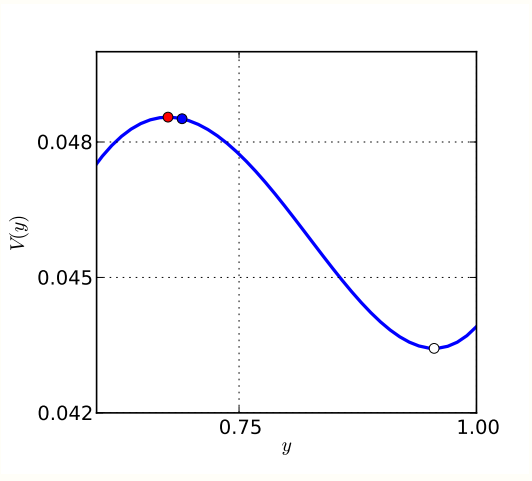

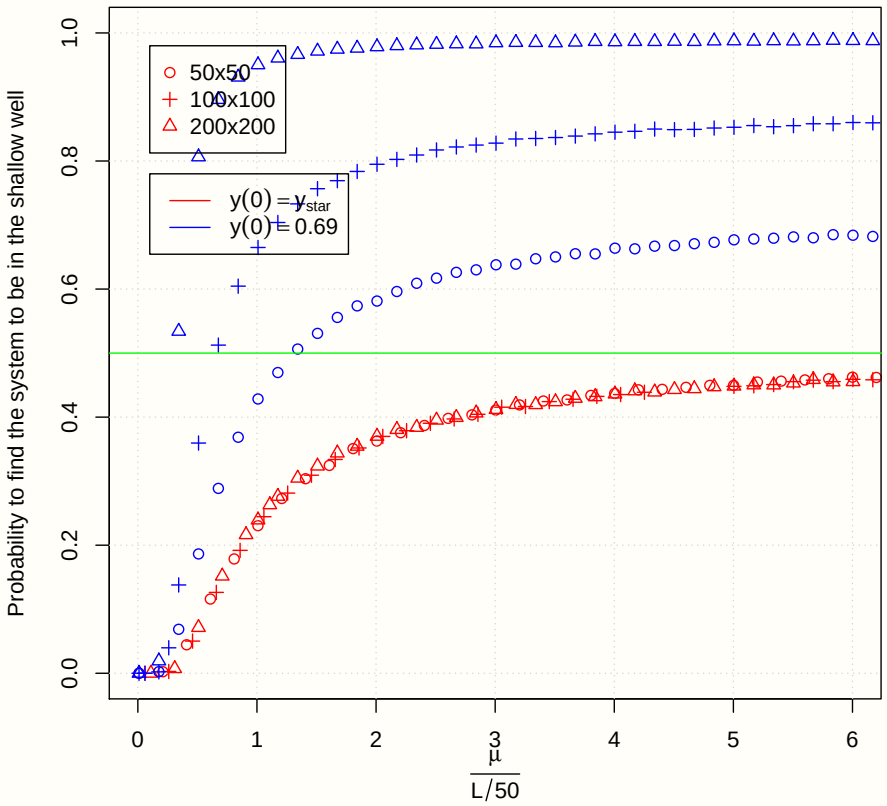

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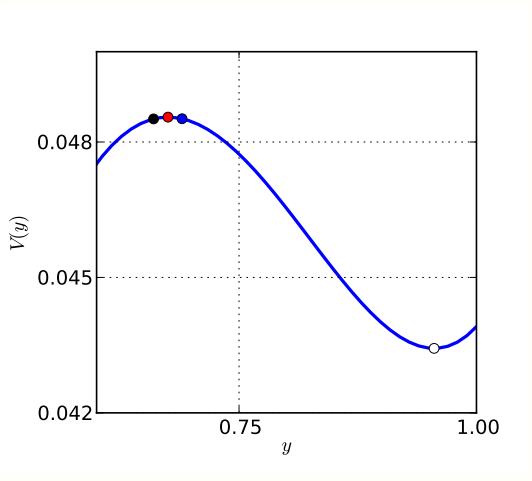

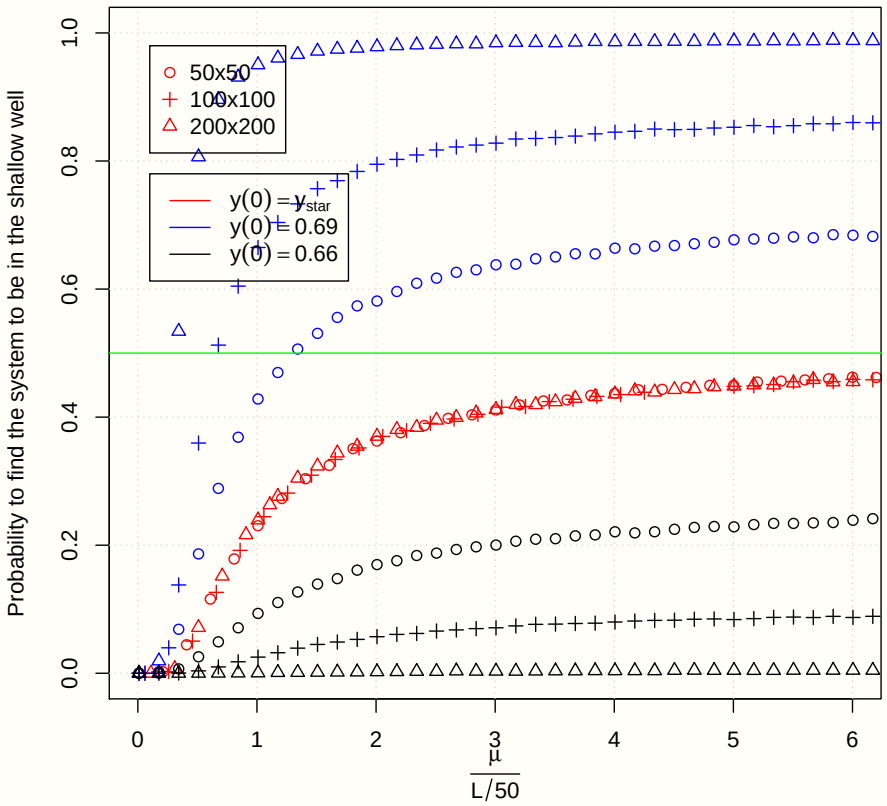

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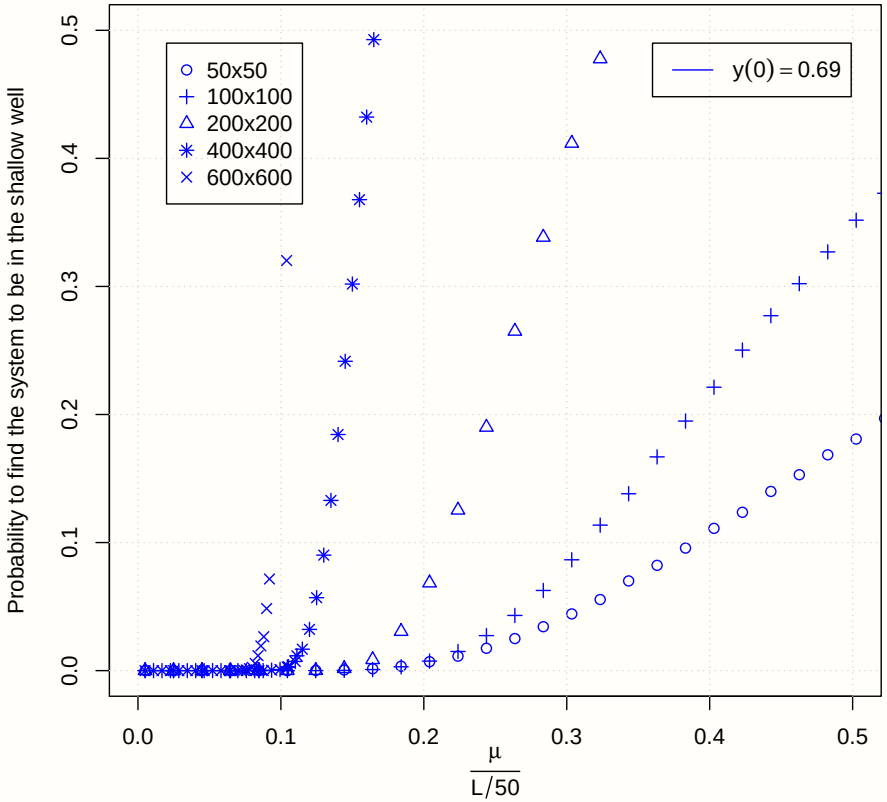
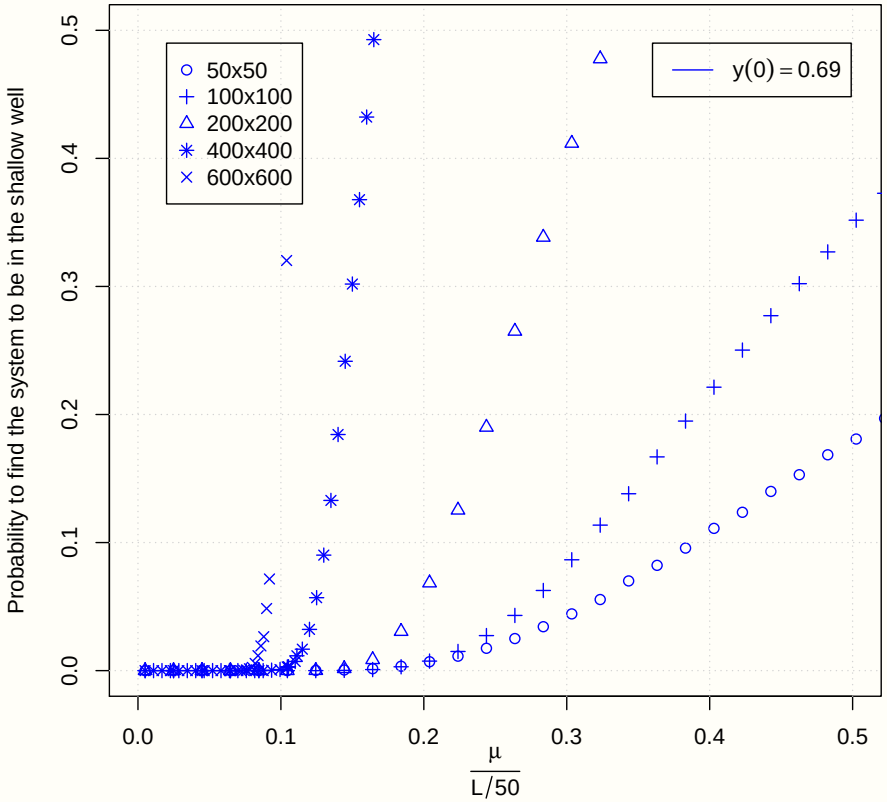
Critical μ/L?



- 2d lattice $L \times L$, nhood 8; thresholds' PDF is Gaussian, mean 0.55, s.d. 0.225; $y(0) = 0.69 > y_\star$
- **What if we revert back to μ?**

Critical μ/L?



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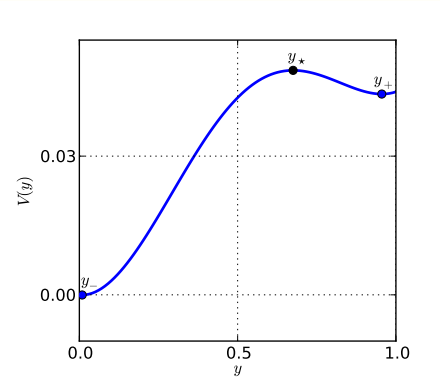

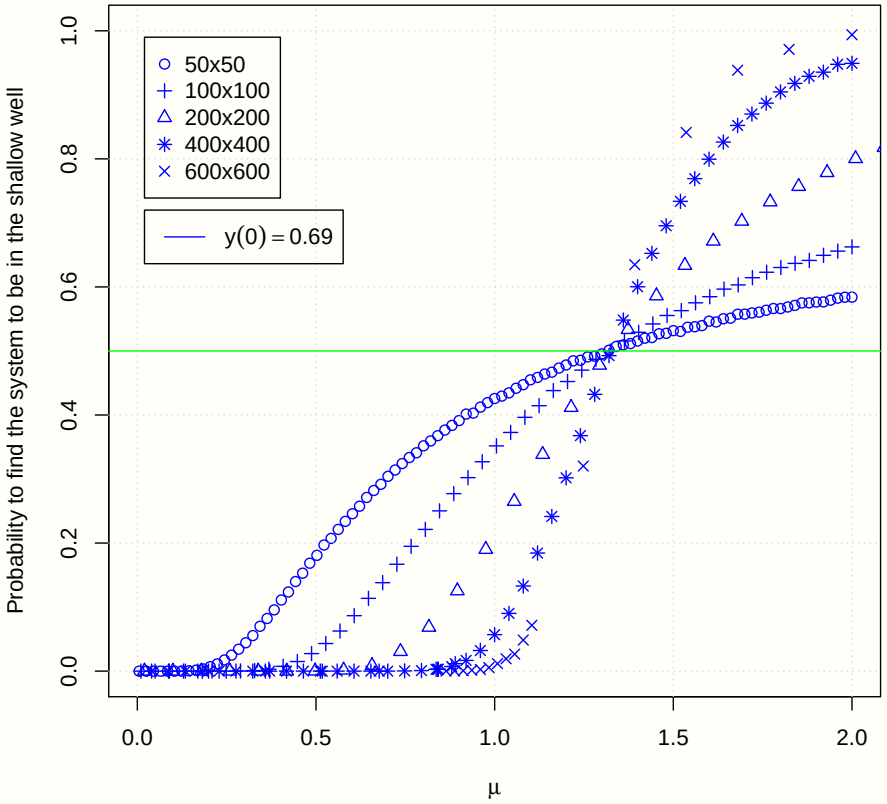

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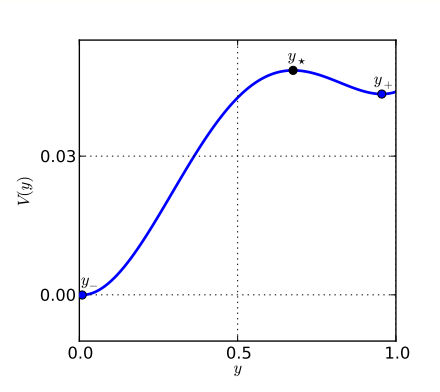

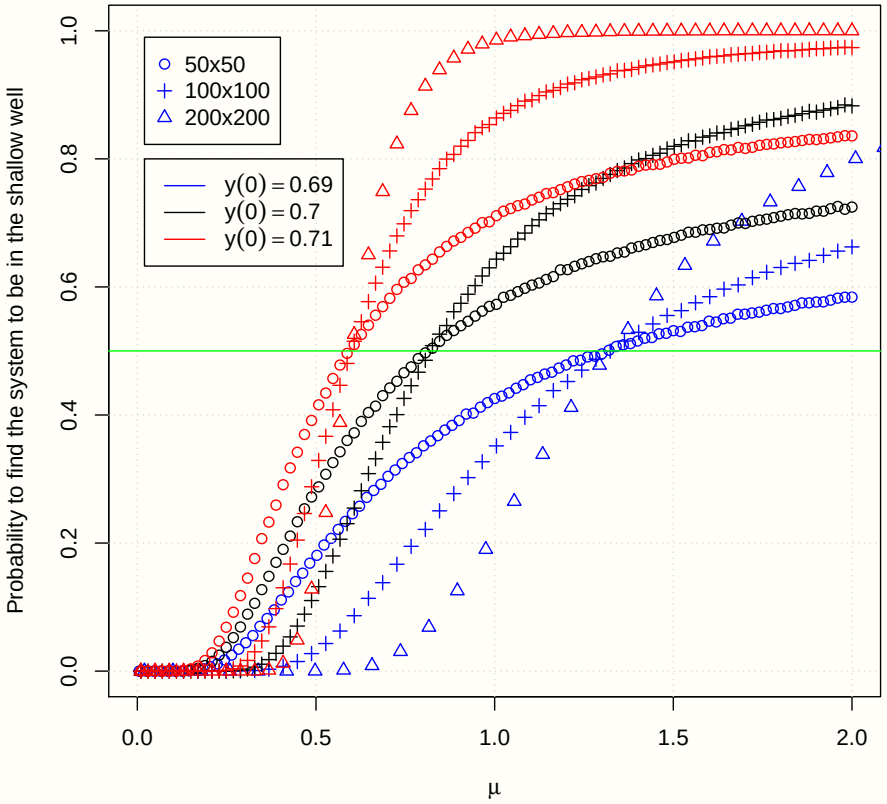

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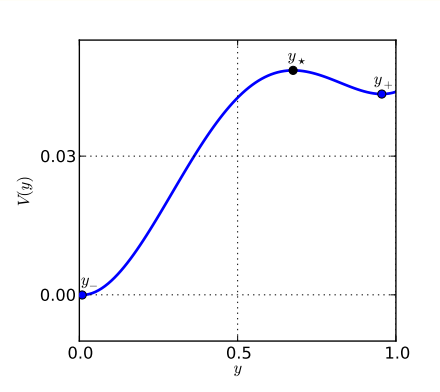

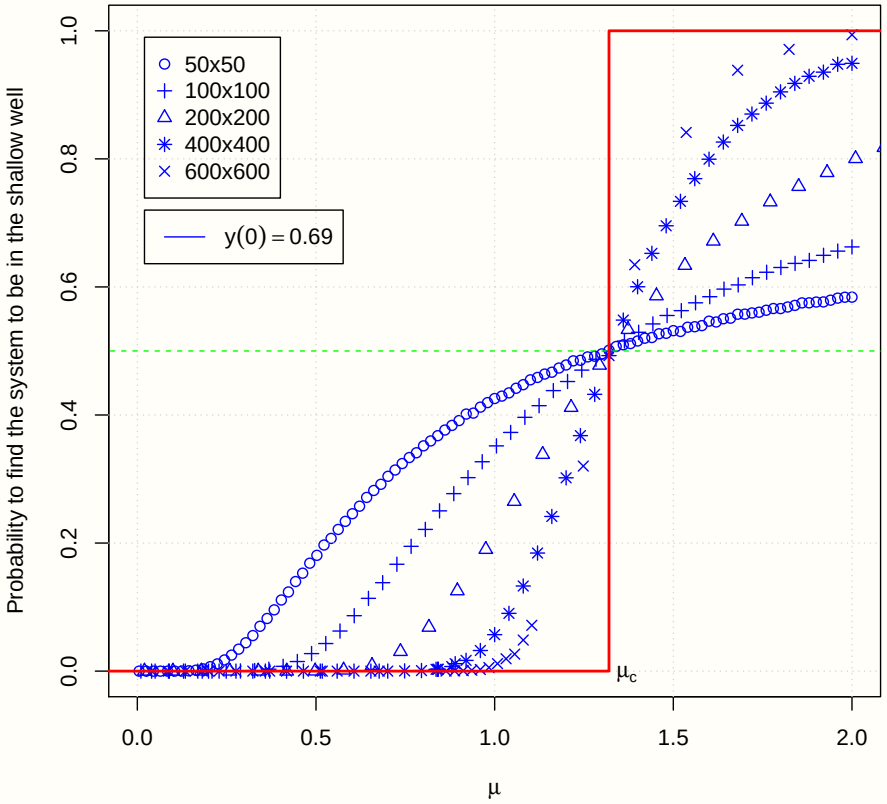

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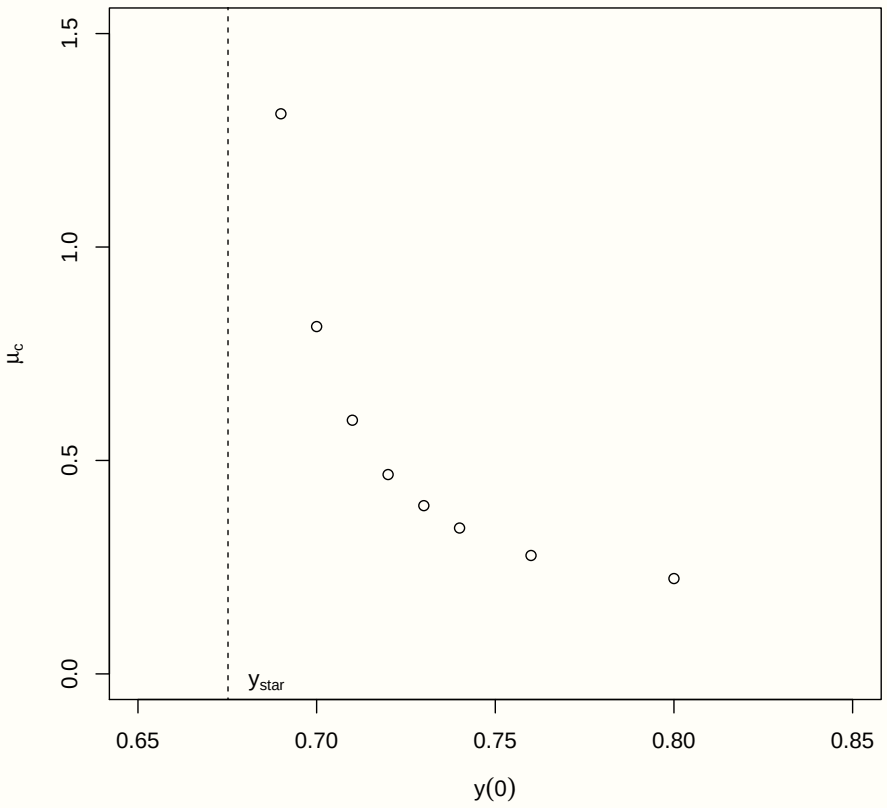
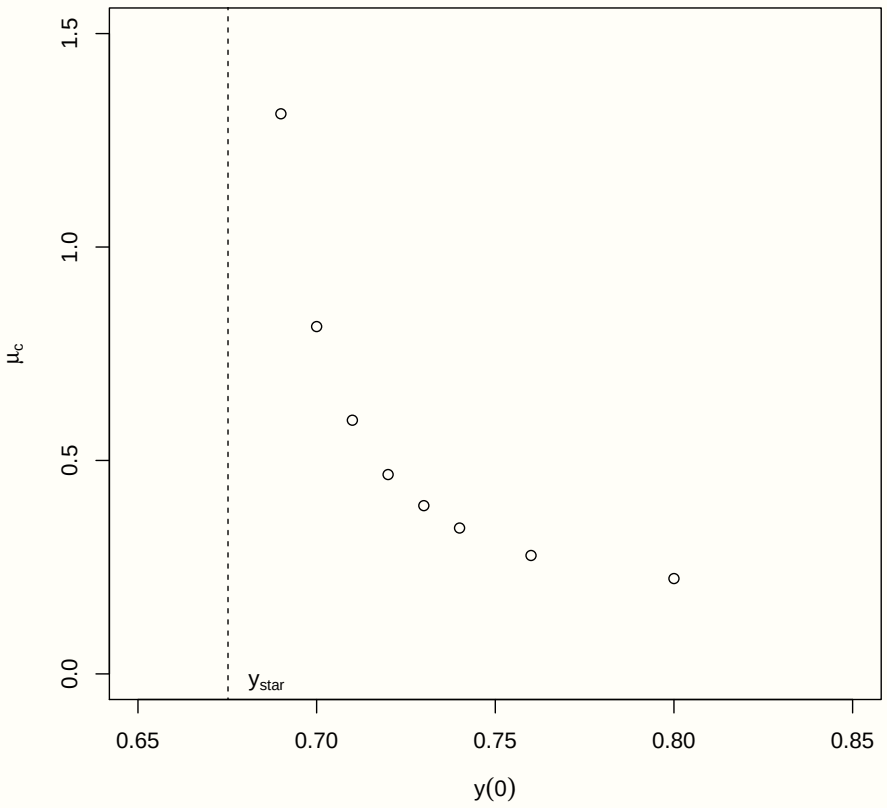
- **In large-scale limit this might be a critical value μ_c**



- **Try to fit with** $\mu_c = \alpha + \frac{\beta}{y(0) - y_\star}$
- **Indeed**, $\alpha \approx 0.0613$ and $\beta \approx 0.0185$
- **The lowest value:** $\mu_\star = \mu_c(1.0) \approx 0.1183$



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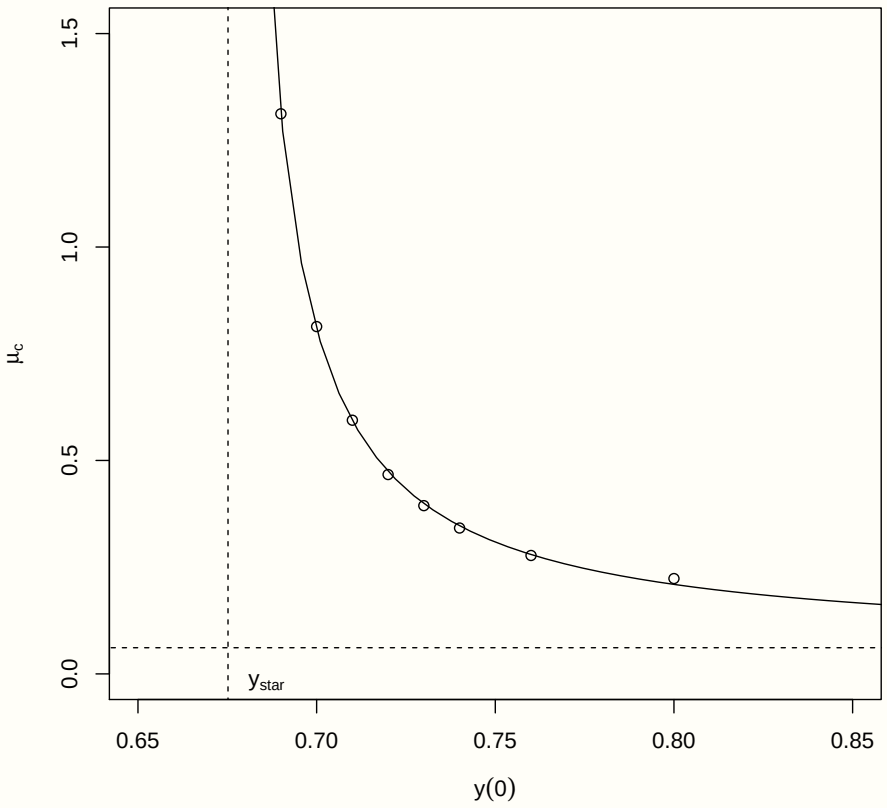

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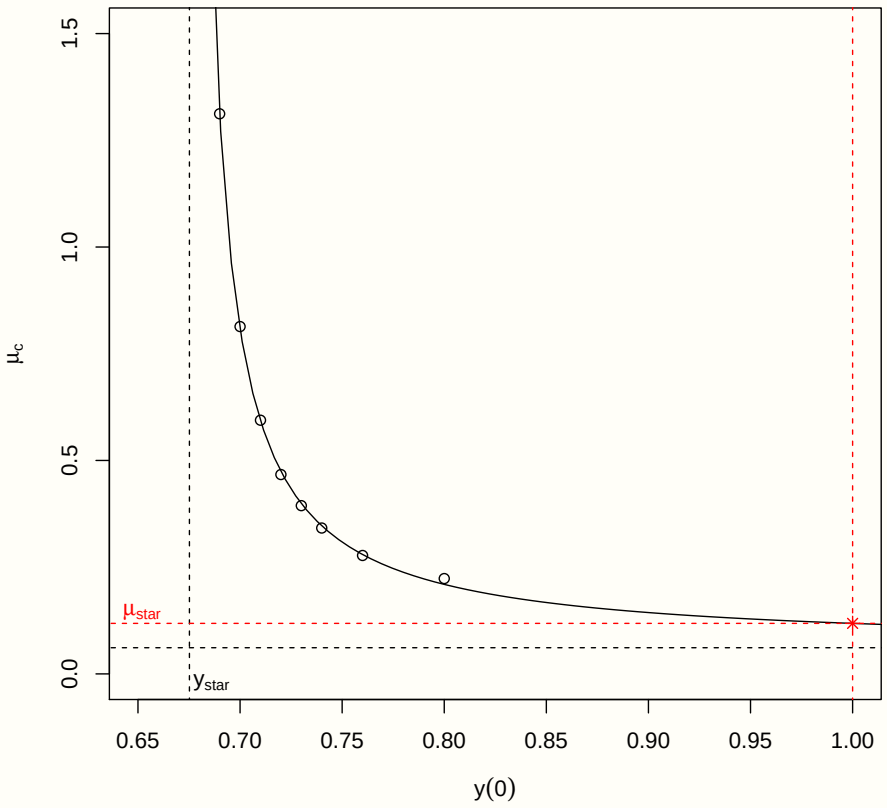

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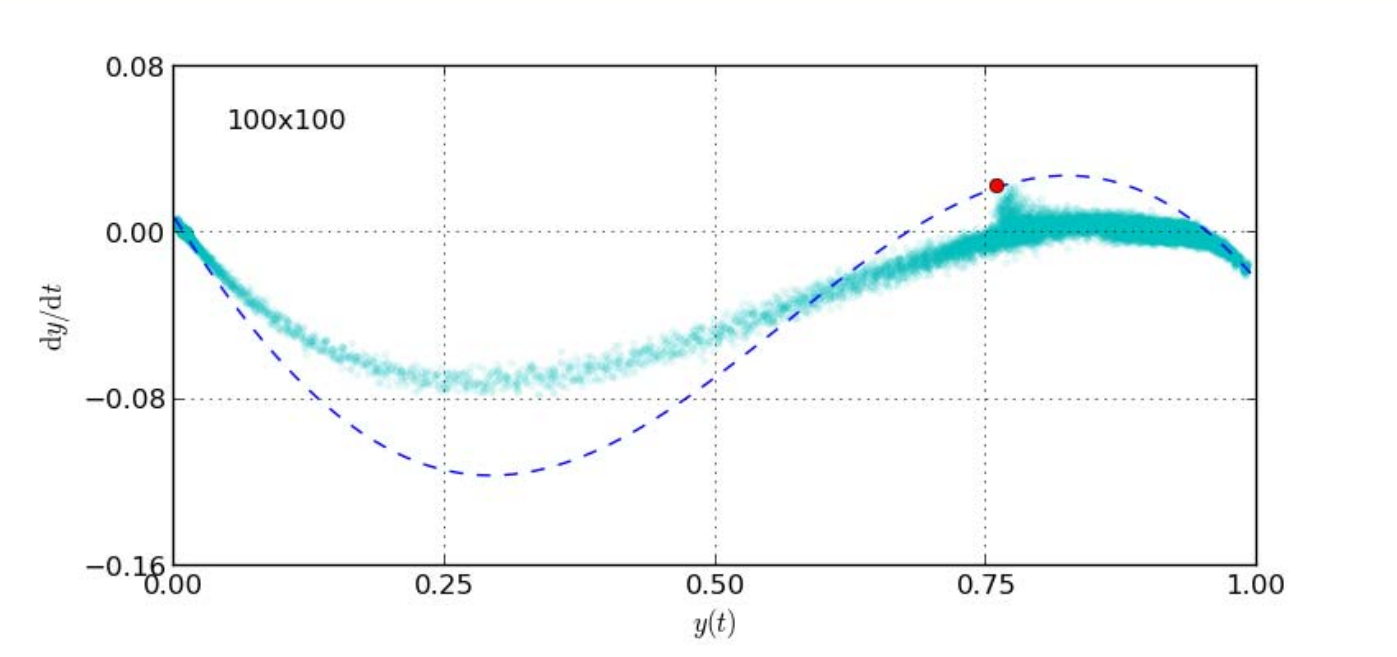
Critical μ & slow manifold

- 2d lattice, nhood 8
- thresholds' PDF is Gaussian, mean 0.55 s.d. 0.225
- **First**, $y(0) = 0.76 \Rightarrow \mu_c \approx 0.2779$



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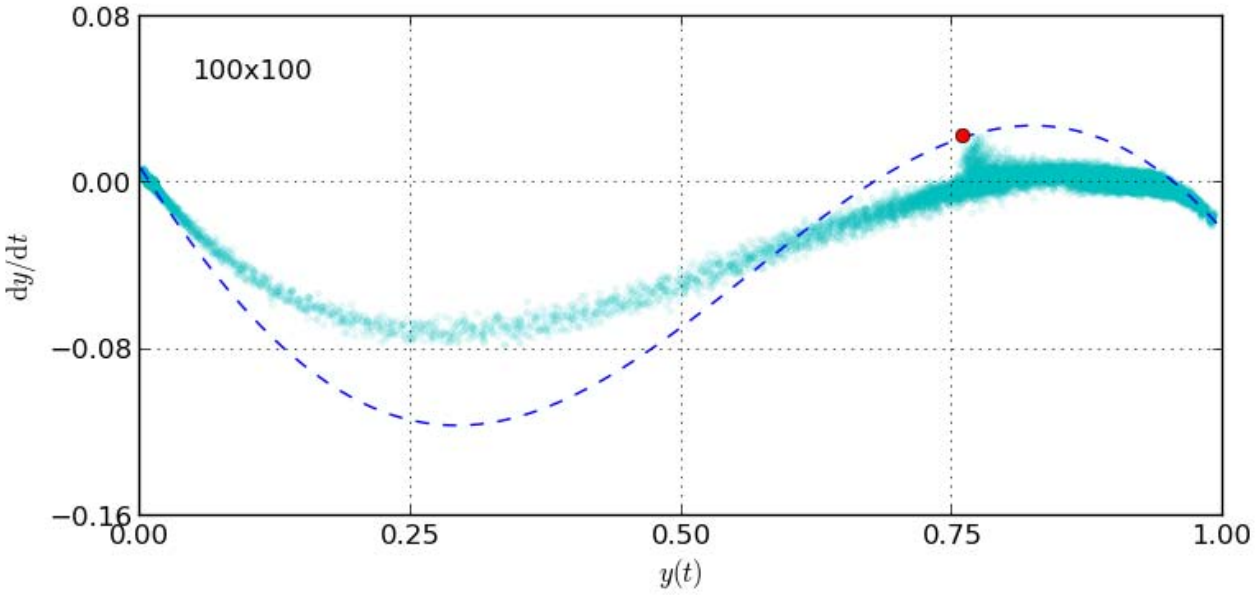

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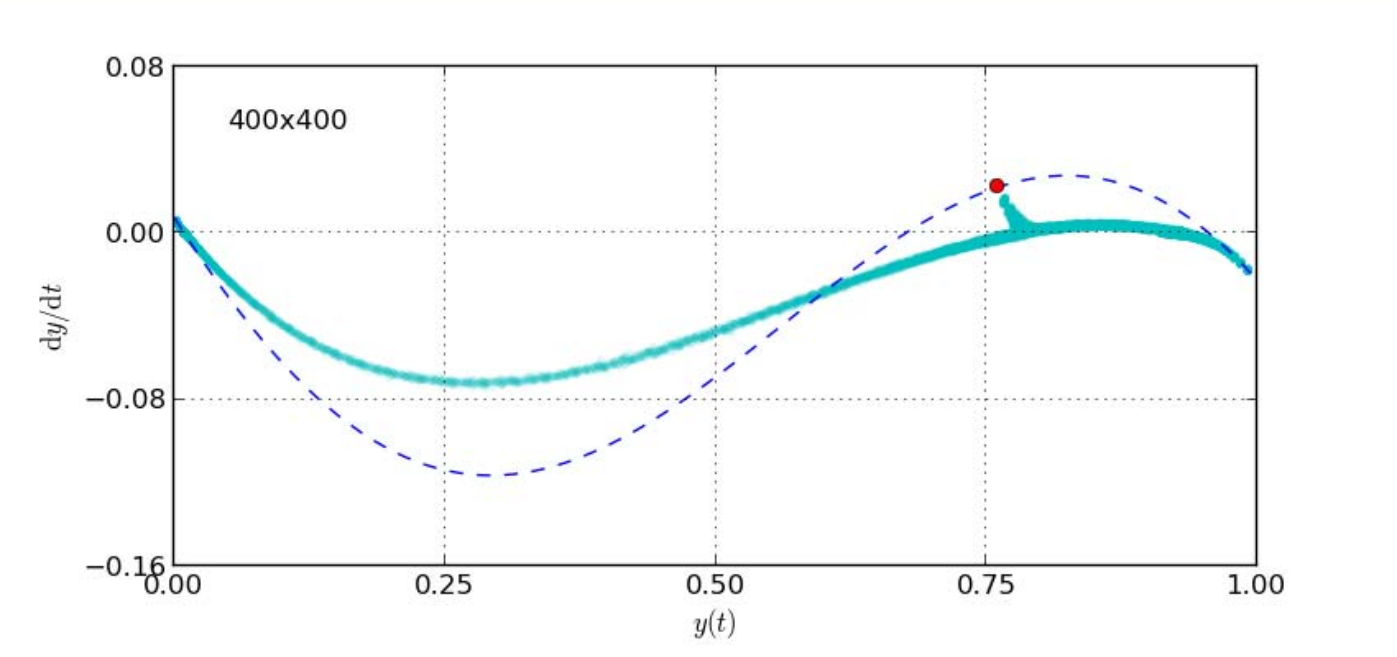

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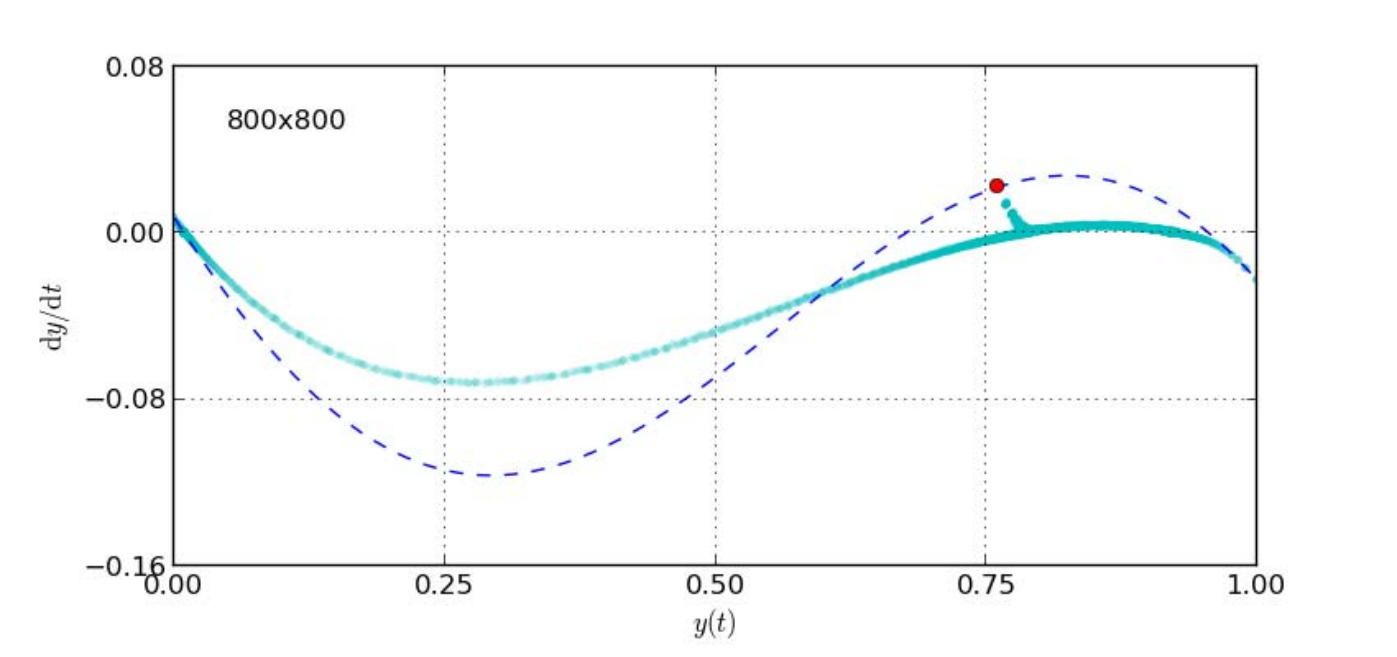
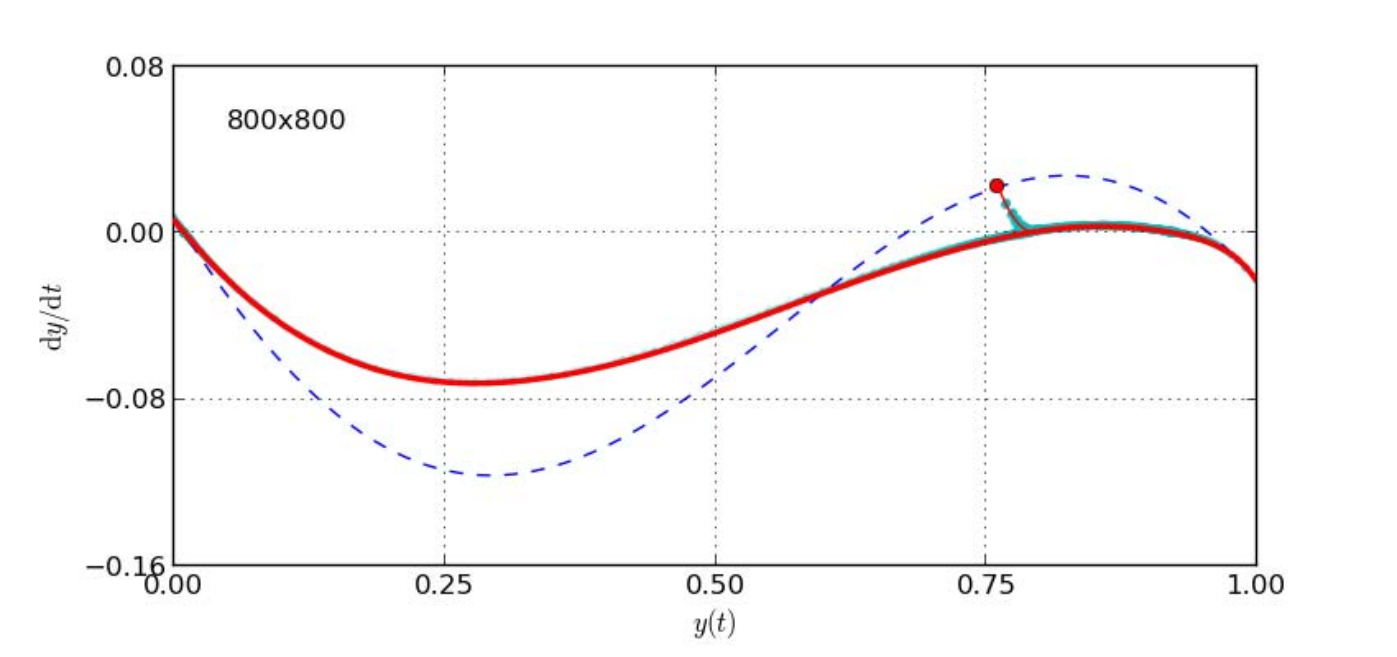


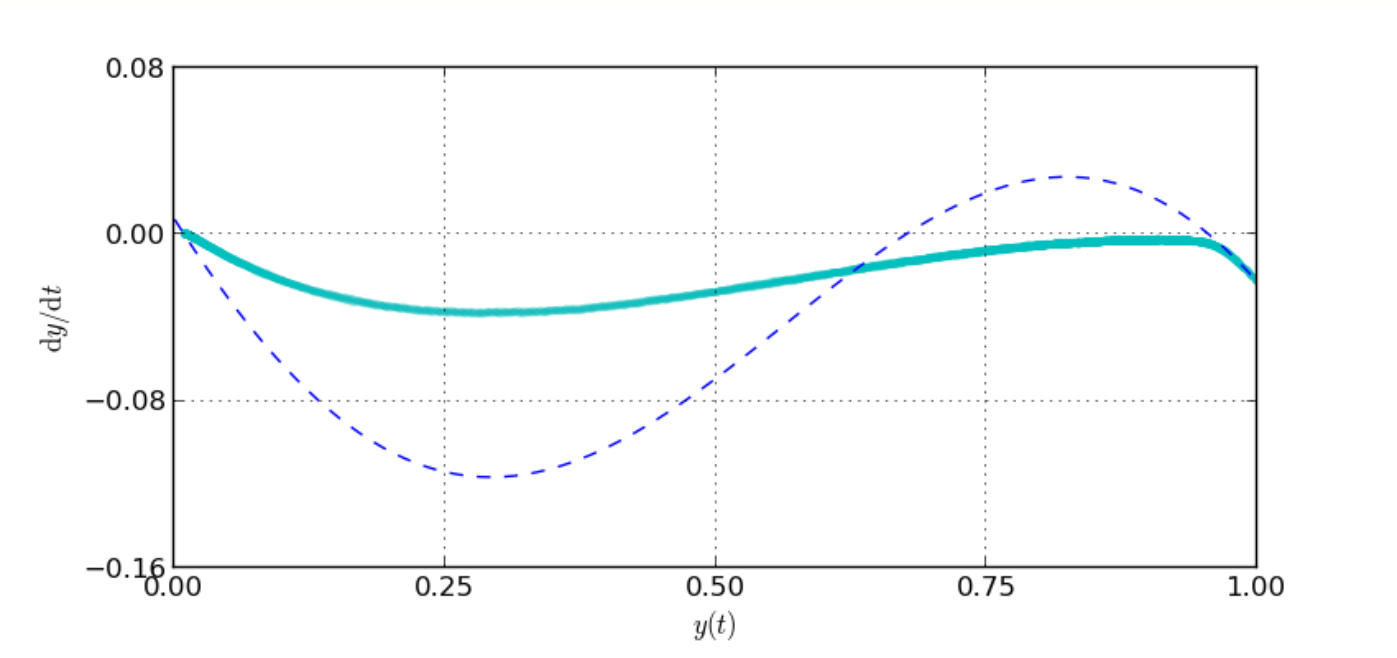
Critical μ & slow manifold

- 2d lattice, nhood 8
- thresholds' PDF is Gaussian, mean 0.55 s.d. 0.225
- $y(0) = 0.76 \Rightarrow \mu_c \approx 0.2779$



Critical μ & slow manifold

- We expect that at $\mu_\star$ the slow-manifold curve becomes tangent to the x axis, and the shallow well vanishes.
- it looks like our estimate's a little low, but the idea's right.



Discussion

- The threshold model for coordination of social norms has all the **complexity** of the Random Field Ising Model
- Outcome can depend sensitively on distribution of thresholds
- **Neighborhood size** can change the outcome, even with very fast mixing
- Relaxation to global minimum is helped by **lower mixing rate**, not higher — unlike temperature
- With finite mixing rates, there is a **"slow manifold"**, which suggests a simplifying analysis, including a potential function

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